Developments in IRS Multi-Mode Waveforms

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Gravitational Waves from Numerical Mergers

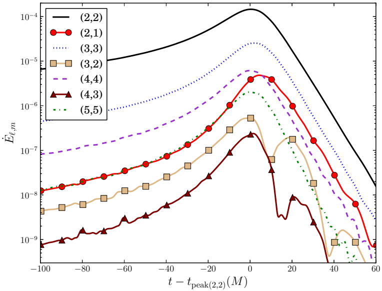
NR black-hole merger simulations produce waveforms decomposed into (spin-weighted) spherical harmonics:

$$r\psi_4(t,r,\theta,\phi) = \sum_{\ell m} C_{\ell m}(t,r)\ {}_{-2}Y_\ell^m(\theta,\phi).$$

- We work with *strain-rate* $\dot{h} = \int \psi_4^*\, dt$.
- Each mode has an amplitude and complex phase:

$$r\dot{h}_{\ell m} = A_{\ell m}(t) e^{i\varphi_{\ell m}(t)}.$$

- A handful of modes dominate energy flux; mostly $(\ell, \pm\ell)$.
- $(2, \pm 2)$ is sufficient for *detection*; other modes are important for *parameter estimation*.



Modal power for 4:1.

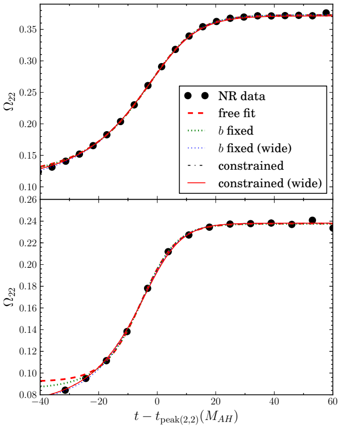
Implicit Rotating Source Picture

[Baker et al. (2008), Kelly et al. (2011)]:

- Most important WF modes had consistent *rotational phases* $\Phi_{\ell m} \equiv \varphi_{\ell m}/m$
- Best matches for $\ell = m$ modes
- Rotational frequency model is a smoothed "step function" to fundamental QNM frequency:

$$
\begin{aligned}
\Omega(t) &= \Omega_f(1 - f(t)), \\
f(t) &\propto \left[1 - \left(1 + \frac{1}{\kappa} e^{-2(t-t_0)/b}\right)^{-\kappa}\right]
\end{aligned}
$$

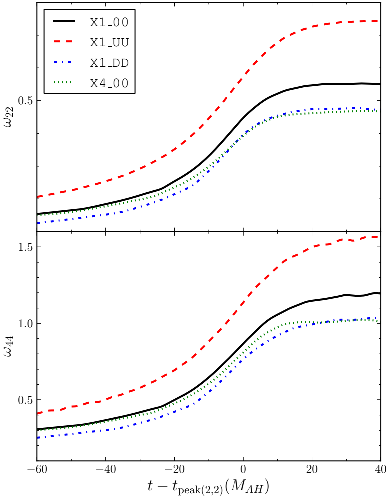
- Poor for times earlier than $\sim 60M$ before merger
- Still need model for individual mode amplitudes



Fit of $\Omega(t)$ for $(2,2)$ mode of `X1_UU` (top) and `X1_DD` (bottom).

Aligned Spins: Frequency Evolution

- Asymptotes to quasinormal mode frequency for $(\ell, \pm\ell)$
- `X4_00` tracks with `X1_00` before merger, `X1_DD` after
- How do `X4_00` & `X1_DD` differ?
- All BHB mergers result in Kerr hole parametrised by mass M_f, spin $a_\mathrm{f} \equiv M_\mathrm{f} j_\mathrm{f} \equiv S_\mathrm{f}/M_\mathrm{f}$.
- Hints at simplest way of breaking QNM degeneracy in waveform models
- Using end-state predictions for aligned-spin mergers, can construct "equivalence classes" of BHBs



$\omega_{\ell m}$ for $m = \ell$ modes.

Equivalence Classes of BHB Mergers

Aligned-spin BHB configurations "equivalent" to nonspinning 1:1, 4:1, and 6:1 BHBs.

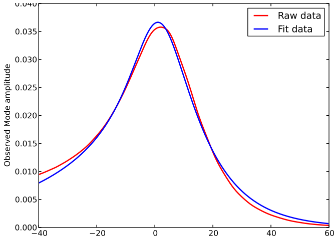
run name	$q \equiv M_1/M_2$	j_1	j_2	j_f
X1_00	1.0	0.0	0.0	0.6865
X1_UD	1.0	0.8	-0.8	0.6865
X5_U0	5.0	0.0937	0.0	0.4748
X4_00	4.0	0.0	0.0	0.4748
X3_D0	3.0	-0.1557	0.0	0.4760
X2_D0	2.0	-0.4734	0.0	0.4765
X1_DD	1.0	-0.6365	-0.6365	0.4765
X6_00	6.0	0.0	0.0	0.3762
X5_D0	5.0	-0.0752	0.0	0.3762
X4_D0	4.0	-0.1896	0.0	0.3762
X3_D0	3.0	-0.3843	0.0	0.3762
X2_DD	2.0	-0.6250	-0.6250	0.3762

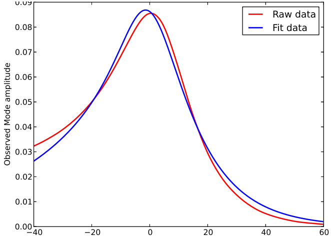
Modeling IRS Multimode Amplitudes

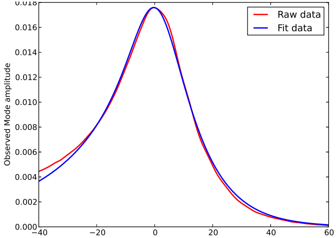
IRS model [Kelly et al. (2011)] suggested a general form for IRS amplitude functions:

$$A_{\ell m}(t) = A_0 \sqrt{\frac{\left|\dot{f}(t)\right|^n}{1 + \alpha_1 \left(f^2 - f^4\right)}}.$$

Three free parameters for each (ℓ, m) pair: A_0, n, α_1.







A_{22} (top), A_{33} (left), A_{44} (bottom) for 4:1 nonspinning merger.

Modeling IRS Multimode Amplitude *Parameters*

- Assemble broad set of aligned-spin BHB mergers.
- Collect amplitude params $\{A_0, n, \alpha_1\}$ over BHB configurations.
- Easier to model cuts along BH parameter directions ...
- Symmetric mass ratio $\eta \equiv M_1 M_2/(M_1 + M_2)^2 \leq 0.25$
- "Total" spin $\tilde{j} \equiv (q^2 j_1 + j_2)/(q^2 + 1)$
- Simplest fit model is product of mass-ratio and spin forms:

$$
\begin{aligned}
A_0(\eta, \tilde{j}) &= g(\eta) \cdot h(\tilde{j}) \\
g(\eta) &= g_0 + g_1(\eta_0 - \eta) + g_2(\eta_0 - \eta)^2, \\
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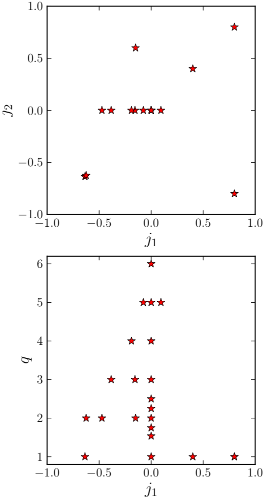

BHB configurations used.

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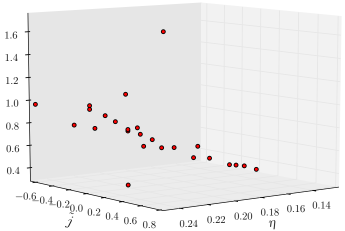

A_0 for (2,2) mode of all configurations.

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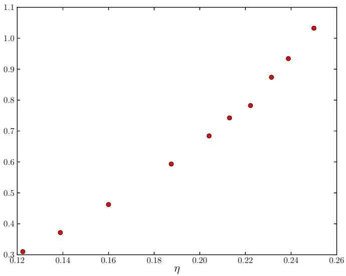

A_0 for (2,2) mode of nonspinning configurations.

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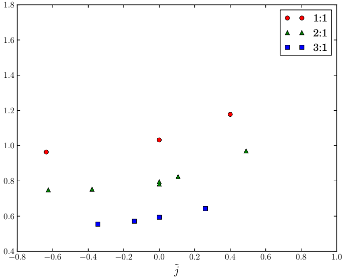

A_0 for (2,2) mode of all 1:1, 2:1, and 3:1 mergers.

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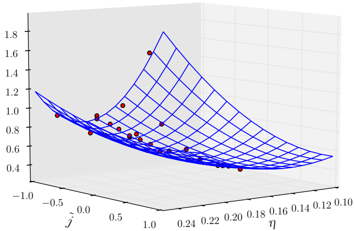


Fit to A_0 for (2,2) mode of all mergers.

Modeling the (3, 2) Mode

Kelly & Baker (2013) showed that "observed" (3, 2) mode at merger is largely (2, 2) mode, leaked through mismatch between *spherical* harmonics ${}_{-2}Y_{\ell'}^{m}$ and *spheroidal* harmonics ${}_{-2}\mathcal{Y}_{\ell}^{m}$.

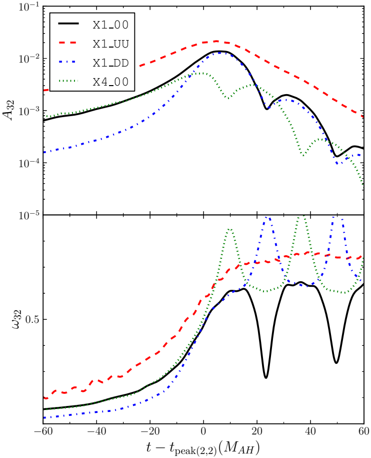
- Visible in both amplitude and frequency.
- QNM eigenfunctions are *spheroidal* harmonics
- Overlap with *spherical* harmonics is

$$s_{\ell'\ell m} = \oint d\Omega \, {}_{-2}\mathcal{Y}_{\ell}^{m}(a_{\mathrm{f}}\sigma_{22}; \theta, \phi) \, {}_{-2}Y_{\ell'}^{m}(\theta, \phi)^*$$

- …leading to mixing coefficients

$$\rho_{\mathrm{basis},\ell 2} \equiv \frac{s_{\ell' 22}}{s_{2' 22}}$$

- Fits observed (3, 2) modes very well; (4, 2) numerics too uncertain



Amplitude and frequency for several runs.

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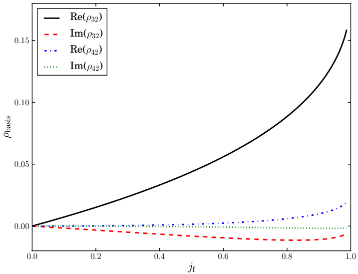

$\rho_{\mathrm{basis},\ell 2}$ over range of final Kerr spins.

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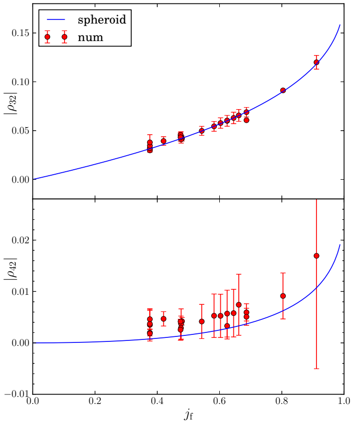

Observed mixing for $(3, 2)$ and $(4, 2)$ modes.

Debumping the $(3, 2)$ Mode

In principle, we can now take the observed $(3, 2)$ mode (red), subtract $(2, 2)$ leakage (blue) to yield beautifully single-peaked “real” $(3, 2)$ modes (green).

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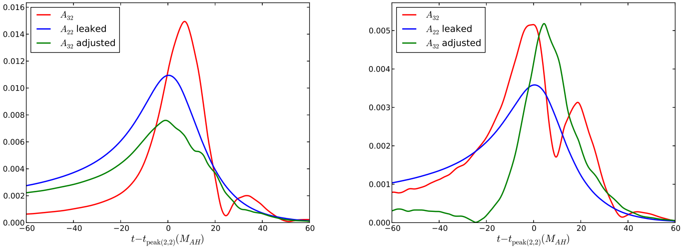

Cleaning $(3, 2)$ modes for `X1_UD` (left) and `X4_00`.

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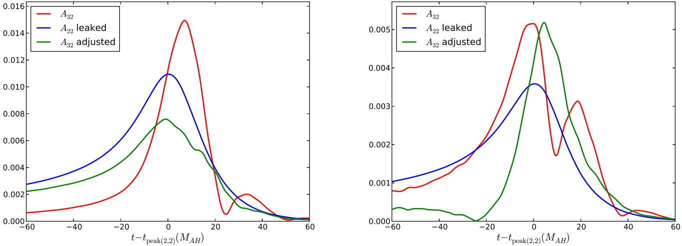

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…not as beautiful as hoped. Other mixing effects may still be important before merger.

Summary

- IRS picture gives useful phase model for late-merger and hybrid waveforms.
- Individual mode amplitudes require separate parameter models (as in post-Newtonian).
- $(\ell, m < \ell)$ modes need to have "leakage" subtracted before fitting.
- *But* more subtraction might be necessary ...
- Next: (1) test against ringdown templates for detection & parameter estimation.
- Next: (2) use higher-accuracy WFs from the NR-AR collaboration to generate better fits.
- (or CCC catalog ...)

Bibliography

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